

COMPACT PRESENTABILITY OF TREE ALMOST AUTOMORPHISM GROUPS

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ABSTRACT. We establish compact presentability, i.e. the locally compact version of finite presentability, for an infinite family of tree almost automorphism groups. As a particular example, we obtain that Neretin's group of spheromorphisms, which is the almost automorphism group of a regular non-rooted tree, is compactly presented.

We additionally obtain an upper bound on the Dehn function of these groups in terms of the Dehn function of an embedded Higman-Thompson's group. This, combined with a result of Guba, implies that the Dehn function of the Neretin group of the regular trivalent tree is polynomially bounded.

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1. INTRODUCTION

Almost automorphism groups. If T is a locally finite tree, then its automorphism group $\text{Aut}(T)$ has a natural locally compact and totally disconnected topology. If moreover T is regular then $\text{Aut}(T)$ acts continuously, properly and cocompactly on T .

More flexible than the notion of automorphism is the notion of almost automorphism. Unlike automorphisms, almost automorphisms do not act on T but on its boundary $\partial_\infty T$. Roughly speaking, an almost automorphism of T is a transformation induced in the boundary by a piecewise tree automorphism. Almost automorphisms form a topological group $\text{AAut}(T)$ containing the automorphism group $\text{Aut}(T)$ as an open subgroup.

In the case where T is a non-rooted regular tree of degree $d+1 \geq 3$, the group \mathcal{N}_d of almost automorphisms of T was introduced by Neretin in connection with his work in representation theory [Ner92]. Neretin proved that from the point of view of representation theory, \mathcal{N}_d can be seen as a p -adic analogue of the diffeomorphism group of the circle. Inspired by a simplicity result of the

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diffeomorphism group of the circle $\text{Diff}^+(\mathbb{S}^1)$ [Her71], Kapoudjian later proved that the group \mathcal{N}_d is abstractly simple [Kap99].

Recently, Bader, Caprace, Gelander and Mozes proved that \mathcal{N}_d does not have any lattice [BCGM12]. This result is remarkable for the reason that all the familiar examples of simple locally compact groups (which are unimodular), e.g. real or p -adic Lie-groups, or the group of type preserving automorphisms of a locally finite regular tree, are known to have lattices. Actually \mathcal{N}_d turned out to be the first example of a locally compact simple group without lattices.

In this paper we investigate a family of groups which appear as generalizations of Neretin's group. Here we give an outline of their construction (see Section 4 for precise definitions). Every finite permutation subgroup $D \leq \text{Sym}(d)$ is known to give rise to a closed subgroup of the automorphism group $\text{Aut}(\mathcal{T}_d)$ of the rooted d -regular tree \mathcal{T}_d , by considering the infinitely iterated permutational wreath product $W(D) = (\dots \wr D) \wr D$. Elements of $W(D)$ are rooted automorphisms whose local action is prescribed by D . We now consider the quasi-regular rooted tree $\mathcal{T}_{d,k}$, in which the root has degree k and other vertices have degree $d + 1$. Roughly, the almost automorphism groups $\text{AAut}_D(\mathcal{T}_{d,k})$ we are interested in, are homeomorphisms of $\partial_\infty \mathcal{T}_{d,k}$ that are piecewise tree automorphisms whose local action is prescribed by D . This family of groups generalizes Neretin's groups because when D is the full permutation group $\text{Sym}(d)$, the group $W(D)$ is the full automorphism group of \mathcal{T}_d and we can check that $\text{AAut}_D(\mathcal{T}_{d,2}) \simeq \mathcal{N}_d$.

These groups appear in [CDM11], where a careful study of the abstract commensurator group of self-replicating profinite wreath branch groups is carried out (we refer to [BEW11] for an introduction to abstract commensurators of profinite groups). Let $W_k(D)$ be the closed subgroup of $\text{Aut}(\mathcal{T}_{d,k})$ fixing pointwise the first level of $\mathcal{T}_{d,k}$ and acting by an element of $W(D)$ in each subtree rooted at level one. Under the additional assumption that $D \leq \text{Sym}(d)$ is transitive and is equal to its normaliser in $\text{Sym}(d)$, the group $\text{AAut}_D(\mathcal{T}_{d,k})$ turns out to be isomorphic to the abstract commensurator group of $W_k(D)$. In particular Neretin's group \mathcal{N}_d is the abstract commensurator group of $W_2(\text{Sym}(d))$, or equivalently the abstract commensurator group of the automorphism group of the non-rooted regular tree of degree $d + 1$.

Our proof that the groups $\text{AAut}_D(\mathcal{T}_{d,k})$ are compactly presented is motivated by their connections with Thompson's groups and their generalizations. Recall that Higman [Hig74] constructed an infinite family of finitely presented infinite simple groups $V_{d,k}$ (sometimes denoted $G_{d,k}$), generalizing the group V introduced by R. Thompson. When $D \leq \text{Sym}(d)$ is the trivial group then $\text{AAut}_D(\mathcal{T}_{d,k})$ is nothing else than $V_{d,k}$ (see Section 3 for details). One of the reasons why combinatorial group theorists became interested in Thompson's groups is because of the combination of simplicity and finiteness properties. Indeed Thompson's groups T and V turned out to be the first known examples of finitely presented infinite simple groups (see [CFP96]). While simplicity results for $\text{AAut}_D(\mathcal{T}_{d,k})$ have recently been obtained in [CDM11], in this paper we settle in the positive the question if whether or not these groups satisfy

the locally compact version of being finitely presented, i.e. being compactly presented.

Compact presentability. Recall that a locally compact group is said to be compactly generated if there exists a compact subset S so that the group generated by S is the whole group G . Less known than the notion of compact generation is the notion of compact presentation. A locally compact group G is said to be compactly presented if it admits a compact generating subset S such that G has a presentation, as an abstract group, with S as set of generators and relators of bounded length (but possibly infinitely many relators). When the group G is discrete, this amounts to saying that G is finitely presented, and like in the discrete case, for a locally compact group, being compactly presented does not depend on the choice of the compact generating set S .

Compact presentability can be interpreted in terms of coarse simple connectedness of the Cayley graph of the group with respect to some compact generating subset. In particular, among compactly generated locally compact groups, being compactly presented is preserved by quasi-isometries. For a proof of this result see for instance [CH].

Our first result is the following:

Theorem 1.1. *For any $k \geq 1, d \geq 2$, and any subgroup $D \leq \text{Sym}(d)$, the group $\text{AAut}_D(\mathcal{T}_{d,k})$ is compactly presented.*

As mentioned earlier, the group $\text{AAut}_D(\mathcal{T}_{d,k})$ contains a dense copy of the Higman-Thompson's finitely presented group $V_{d,k}$. Here we insist on the fact that for a locally compact group, although having a dense finitely generated subgroup is a sufficient condition for being compactly generated, this does not hold for compact presentation, i.e. having a dense finitely presented subgroup does not imply compact presentation of the ambient group. For example, for any non-Archimedean local field \mathbb{K} , the group $\mathbb{K}^2 \rtimes \text{SL}_2(\mathbb{K})$ has a central extension with non-compactly generated kernel, and is therefore not compactly presented (see for instance [CH, Proposition 8.A.23]). However the reader can check that this group admits dense finitely generated free subgroups.

We also emphasize the fact that for the case of Neretin's group, Theorem 1.1 cannot be obtained by proving finite presentation of a discrete cocompact subgroup because these do not exist [BCGM12]. However we note that it seems to be unknown whether Neretin's group \mathcal{N}_d is quasi-isometric to a finitely generated group.

As a by-product of Theorem 1.1 and the main result of [BCGM12], we also obtain that locally compact simple groups without lattices also exist in the realm of compactly presented groups.

Dehn function. Having obtained compact presentability of a locally compact group G naturally leads to the study of an invariant of G , having both geometric and combinatorial flavors, called the Dehn function of G .

From the geometric point of view, the Dehn function $\delta_G(n)$ is the supremum of areas of loops in G of length at most n . In other words, it is the best isoperimetric function, where isoperimetric function can be understood like for simply connected Riemannian manifolds.

From the combinatorial perspective, the Dehn function is a quantified version of compact presentability: $\delta_G(n)$ is the supremum over all relations w of length at most n in the group, of the minimal number of relators needed to convert w to the trivial word.

First recall that for any two functions $f, g : \mathbb{N} \rightarrow \mathbb{N}$, f is asymptotically bounded by g , which is denoted by $f \preceq g$, if for some constant c we have $f(n) \leq cg(cn) + cn + c$ for every $n \geq 0$; and f, g have the same \approx -asymptotic behavior, denoted by $f \approx g$, if $f \preceq g$ and $g \preceq f$.

If G is compactly presented and if S is a compact generating set, then for some $k \geq 1$ the group G has the presentation $\langle S \mid R_k \rangle$, where R_k is set of relations in G of length at most k . The area $a(w)$ of a relation w , i.e. a word in the letters of S which represents the identity in G , is the smallest integer m so that w can be written in the free group F_S as a product of m conjugates of relators of R_k . Now define the Dehn function of G by

$$\delta_G(n) = \sup \{a(w) : w \text{ relation of length at most } n\}.$$

This function depends on the choice of S and k , but its \approx -asymptotic behavior does not, and is actually a quasi-isometry invariant of G .

Our second result is the following upper bound on the Dehn function of almost automorphism groups:

Theorem 1.2. *For any $k \geq 1, d \geq 2$, and any subgroup $D \leq \text{Sym}(d)$, the Dehn function of $\text{AAut}_D(\mathcal{T}_{d,k})$ is asymptotically bounded by that of $V_{d,k}$.*

On the other hand, the Dehn function of $\text{AAut}_D(\mathcal{T}_{d,k})$ is not linear because having a linear Dehn function characterizes Gromov-hyperbolic groups among compactly presented groups, and the group $\text{AAut}_D(\mathcal{T}_{d,k})$ is easily seen not to be Gromov-hyperbolic. So by a general argument (see for example [Bow95]), the Dehn function of $\text{AAut}_D(\mathcal{T}_{d,k})$ has a quadratic lower bound.

In the case $d = 2$, all the groups $V_{2,k}$ turn out to be isomorphic to Thompson's group V . While the Dehn function of Thompson's group F has been proved to be quadratic [Gub06], it is not known whether the Dehn function of V is quadratic or not. However, using a result of Guba [Gub00] who showed the upper bound $\delta_V \preceq n^{11}$, we obtain:

Corollary 1.3. *Neretin's group \mathcal{N}_2 has a polynomially bounded Dehn function ($\preceq n^{11}$).*

We believe that the result of Guba could be extended to the family of groups $V_{d,k}$, i.e. that every group $V_{d,k}$ satisfies a polynomial isoperimetric inequality. By Theorem 1.2 this would imply that the Dehn function of $\text{AAut}_D(\mathcal{T}_{d,k})$ is polynomially bounded for arbitrary $k \geq 1, d \geq 2$ and $D \leq \text{Sym}(d)$.

Organization. We start by providing a brief introduction to respectively almost automorphisms of trees and Higman-Thompson's groups in the next two sections. In Section 4 we define the groups $\text{AAut}_D(\mathcal{T}_{d,k})$ and their topology, and establish some preliminary results. Section 5, which is the core of the paper, contains the proofs of Theorem 1.1 and Theorem 1.2.

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2. TREE ALMOST AUTOMORPHISMS

2.1. The quasi-regular rooted tree $\mathcal{T}_{d,k}$ and its boundary. Let A and B be finite sets of cardinality respectively $k \geq 1$ and $d \geq 2$. Consider the set of finite words $\{\emptyset\} \cup \{ab_1 \cdots b_n : a \in A, b_i \in B\}$ over the alphabet $X = A \cup B$ being either empty or beginning by an element of A . This set is naturally the vertex set of a rooted tree, where the root is the empty word \emptyset and two vertices are adjacent if they are of the form v and vx , $x \in X$. We will denote this tree by $\mathcal{T}_{d,k}$. In the case when $k = d$ it will be denoted by \mathcal{T}_d . For any vertex v , we will also denote by $\mathcal{T}_{d,k}^v$ the subtree of $\mathcal{T}_{d,k}$ spanned by vertices having v as a prefix. The distance between a vertex and the root will be called its level, and the number of its neighbours will be called its degree. If v is a vertex of level $n \geq 0$, then its neighbours of level $n + 1$ are called the descendants of v . By construction, the root of $\mathcal{T}_{d,k}$ has degree k , and a vertex of level $n \geq 1$ has degree $d + 1$: it has one distinguished neighbour pointing toward the root, and d descendants. See Figure 2.1 for the case $k = 2$, $d = 3$.

The boundary $\partial_\infty \mathcal{T}_{d,k}$ of the tree $\mathcal{T}_{d,k}$ is defined as the set of infinite words $ab_1 \cdots b_n \cdots$, i.e. infinite geodesic rays in $\mathcal{T}_{d,k}$ started at the root. We define the distance between two such words ξ, ξ' by $d(\xi, \xi') = d^{-|\xi \wedge \xi'|}$, where $|\xi \wedge \xi'|$ is the length of the longest common prefix of ξ and ξ' . Equipped with this distance, the boundary at infinity $\partial_\infty \mathcal{T}_{d,k}$ turns out to be homeomorphic to the Cantor set.

From now and for the rest of the paper, we fix an embedding of $\mathcal{T}_{d,k}$ in the oriented plane. This embedding induces a canonical way of ordering, say from left to right, the descendants of any vertex. In particular we obtain a total ordering on the boundary at infinity $\partial_\infty \mathcal{T}_{d,k}$, defined by declaring that $\xi \leq \xi'$ if the first letter of ξ following the longest common prefix of ξ and ξ' , is smaller than the one of ξ' .

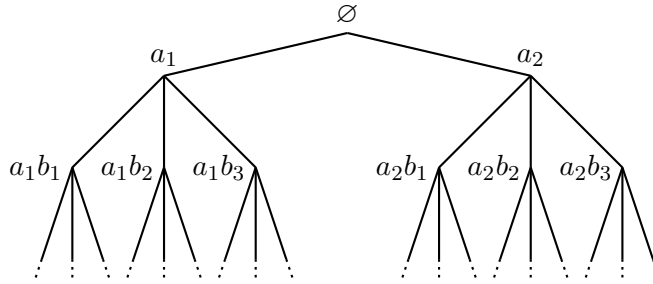


FIGURE 1. A picture of the tree $\mathcal{T}_{3,2}$.

2.2. The group $\text{AAut}(\mathcal{T}_{d,k})$ of almost automorphisms of $\mathcal{T}_{d,k}$. Recall that the group $\text{Aut}(\mathcal{T}_{d,k})$ of automorphisms of the rooted tree $\mathcal{T}_{d,k}$ is defined as the

group of bijections of the set of vertices fixing the root and preserving the edges. In particular every automorphism of $\mathcal{T}_{d,k}$ induces a homeomorphism of $\partial_\infty \mathcal{T}_{d,k}$. We now introduce a larger subgroup of the homeomorphism group of $\partial_\infty \mathcal{T}_{d,k}$, namely the group of homeomorphisms of $\partial_\infty \mathcal{T}_{d,k}$ which are piecewise tree automorphisms.

Definition 2.1. A finite subtree T of $\mathcal{T}_{d,k}$ is a rooted complete subtree if it contains the root as a vertex of degree k and if any other vertex which is not a leaf has degree $d + 1$.

If T is a finite rooted complete subtree of $\mathcal{T}_{d,k}$ then its complement is a forest composed of finitely many copies of the rooted d -regular tree \mathcal{T}_d . If T, T' are subtrees of $\mathcal{T}_{d,k}$, a map $\psi : \mathcal{T}_{d,k} \setminus T \rightarrow \mathcal{T}_{d,k} \setminus T'$ will be called a forest isomorphism if it maps each connected component of $\mathcal{T}_{d,k} \setminus T$ to a connected component of $\mathcal{T}_{d,k} \setminus T'$, and induces a tree isomorphism on each of these connected components. Note that a forest isomorphism ψ naturally induces a homeomorphism $\tilde{\psi}$ of the boundary $\partial_\infty \mathcal{T}_{d,k}$.

Definition 2.2. The group $\text{AAut}(\mathcal{T}_{d,k})$ is defined as the set of equivalence classes of triples (ψ, T, T') , where T, T' are finite rooted complete subtrees such that $|\partial T| = |\partial T'|$ and $\psi : \mathcal{T}_{d,k} \setminus T \rightarrow \mathcal{T}_{d,k} \setminus T'$ is a forest isomorphism, where two triples $(\psi_1, T_1, T'_1), (\psi_2, T_2, T'_2)$ are said to be equivalent if $\tilde{\psi}_1 = \tilde{\psi}_2$. The multiplication in $\text{AAut}(\mathcal{T}_{d,k})$ is inherited from the composition in $\text{Homeo}(\partial_\infty \mathcal{T}_{d,k})$.

We mention the following result, whose proof is easy and left to the reader, which gives an alternative definition of the group of almost automorphisms $\text{AAut}(\mathcal{T}_{d,k})$.

Lemma 2.3. *Let T_1, T'_1, T_2, T'_2 be finite complete rooted subtrees of $\mathcal{T}_{d,k}$. Then two triples $(\psi_1, T_1, T'_1), (\psi_2, T_2, T'_2)$ are equivalent if and only if there exist finite rooted complete subtrees T, T' so that T (resp. T') contains both T_1 and T_2 (resp. T'_1 and T'_2) and $\psi_1, \psi_2 : \mathcal{T}_{d,k} \setminus T \rightarrow \mathcal{T}_{d,k} \setminus T'$ are equal.*

Remark 2.4. By the previous lemma, when considering a triple (ψ, T, T') representing an element of $\text{AAut}(\mathcal{T}_{d,k})$, we can always assume that T and T' both contain a given finite subtree of $\mathcal{T}_{d,k}$.

Note that since the only automorphism of $\mathcal{T}_{d,k}$ inducing the trivial homeomorphism on $\partial_\infty \mathcal{T}_{d,k}$ is the identity, the group $\text{AAut}(\mathcal{T}_{d,k})$ contains a copy of the group $\text{Aut}(\mathcal{T}_{d,k})$ of automorphisms of the tree $\mathcal{T}_{d,k}$.

3. HIGMAN-THOMPSON'S GROUPS

3.1. Introduction. R. Thompson introduced in 1965 three groups $F \leq T \leq V$, an introduction to which can be found in [CFP96], while constructing a finitely generated group with unsolvable word problem. The groups T and V turned out to be the first examples of finitely presented infinite simple groups. Higman then generalized Thompson's group V to an infinite family of groups (which were originally denoted by $G_{d,k}$, but we will use the notation $V_{d,k}$ to keep in mind the analogy with Thompson's group V , which is nothing else

than $V_{2,1}$). K. Brown later generalized Higman's construction to an infinite family of groups $F_{d,k} \leq T_{d,k} \leq V_{d,k}$ such that $F_{2,1} \simeq F$ and $T_{2,1} \simeq T$.

These groups were originally defined as automorphism groups of certain free algebras. We refer the reader to [Bro87] for an introduction from this point of view.

The definition of the groups $F_{d,k}, T_{d,k}, V_{d,k}$ we give below is in term of homeomorphism groups of the boundary of the quasi-regular rooted tree $\mathcal{T}_{d,k}$. From this point of view, elements of these groups can be represented either as homeomorphisms of $\partial_\infty \mathcal{T}_{d,k}$ or by combinatorial diagrams, and we will use the interplay between these two representations.

3.2. Higman-Thompson's groups $V_{d,k}$ as subgroups of $\text{AAut}(\mathcal{T}_{d,k})$. The point of view which is adopted here to define Higman-Thompson's groups $V_{d,k}$ is mostly borrowed from [CDM11].

Definition 3.1. An element of $\text{AAut}(\mathcal{T}_{d,k})$ is called locally order-preserving if it can be represented by a triple (ψ, T, T') such that T, T' are complete rooted subtrees of $\mathcal{T}_{d,k}$ and $\psi : \mathcal{T}_{d,k} \setminus T \rightarrow \mathcal{T}_{d,k} \setminus T'$ preserves the order of the boundary at infinity on each connected component.

It follows from the order-preserving condition that such a forest isomorphism ψ is uniquely determined by the induced bijection between the leaves of T and the leaves of T' . Locally order-preserving almost automorphisms are easily checked to form a subgroup of $\text{AAut}(\mathcal{T}_{d,k})$.

Definition 3.2. The Higman-Thompson's group $V_{d,k}$ is defined as the subgroup of $\text{AAut}(\mathcal{T}_{d,k})$ of locally order-preserving elements.

Every locally order-preserving $v \in V_{d,k}$ has a unique representative (ψ, T, T') so that T, T' are complete rooted subtrees of $\mathcal{T}_{d,k}$ and T is minimal for the inclusion. Then T' is also minimal and (ψ, T, T') will be called the canonical representative of v . This notion coincides with the classical notion of reduced tree pair diagrams commonly used to study Thompson's groups. The tree T will be called the domain tree of v and T' the range tree. When considering a triple representing a locally order-preserving element, we will without further mention assume that this is the canonical representative.

The planarity of $\mathcal{T}_{d,k}$ induces a canonical way of ordering the leaves, say from left to right, of any finite rooted complete subtree of $\mathcal{T}_{d,k}$. Following K. Brown, we let $F_{d,k}$ (resp. $T_{d,k}$) be the subgroup of $V_{d,k}$ consisting of elements whose canonical representative (ψ, T, T') is such that $\psi : \mathcal{T}_{d,k} \setminus T \rightarrow \mathcal{T}_{d,k} \setminus T'$ preserves the order (resp. cyclic order) of the leaves.

The following finiteness result is due to Higman [Hig74] (see also [Bro87]).

Theorem 3.3. *Higman-Thompson's groups $V_{d,k}$ are finitely presented.*

This result will be used in Section 5, where we will enlarge a finite presentation of $V_{d,k}$ to obtain a compact presentation of the group $\text{AAut}_D(\mathcal{T}_{d,k})$.

3.3. Saturated subsets. We now introduce a notion of saturated subsets inside the group $V_{d,k}$, needed in Section 5. We would like to point out that this notion is not necessary if one just wants to prove Theorem 1.1. However

it will be used in the proof of Theorem 1.2 to perform the cost estimates carefully.

Definition 3.4. A subset $\Sigma \leq V_{d,k}$ is said to be saturated if for every $\sigma = (\psi, T, T') \in \Sigma$ and every $u \in \text{Aut}(\mathcal{T}_{d,k})$, all the elements of $V_{d,k}$ of the form $(\psi', u(T), T')$ belong to Σ .

Lemma 3.5. *Every finite subset $\Sigma \leq V_{d,k}$ is contained in a finite saturated subset.*

Proof. Let Σ' be the subset of $V_{d,k}$ consisting of elements of the form $(\psi, u(T), T')$, where $u \in \text{Aut}(\mathcal{T}_{d,k})$ and T is the domain tree of the canonical representative of some element of Σ . Clearly Σ' contains Σ and is saturated. Since Σ is finite, the number of such trees T is finite, and so is the set of $u(T)$, $u \in \text{Aut}(\mathcal{T}_{d,k})$. The result then follows from the following observation: if T is a fixed finite complete rooted subtree of $\mathcal{T}_{d,k}$, then there are only finitely many elements of $V_{d,k}$ having a canonical representative of the form (ψ, T, T') . \square

3.4. A lower bound for the word metric in $V_{d,k}$. Here we give a lower bound for the word metric in the group $V_{d,k}$ in terms of a combinatorial data contained in the diagrams (ψ, T, T') representing elements of $V_{d,k}$.

Recall that a d -caret (or caret for short) in $\mathcal{T}_{d,k}$ is a subtree spanned by a vertex of level $n \geq 1$ and its d neighbours of level $n + 1$. We insist on the fact that we do not consider the subtree of $\mathcal{T}_{d,k}$ spanned by the root and its k neighbours as a caret. If T is a finite complete rooted subtree of $\mathcal{T}_{d,k}$ with κ carets, then the number of leaves of T is $(d - 1)\kappa + k$. In particular if $v \in V_{d,k}$ has canonical representative (ψ, T, T') , then T, T' have the same number of leaves, and consequently they also have the same number of carets. By abuse we will call it the number of carets of v and denote it by $\kappa(v)$.

Metric properties of Higman-Thompson's groups of type F and T can be essentially understood in terms of the number of carets of tree diagrams, the latter being quasi-isometric to the word-length associated to some finite generating set. The use of this point of view shed light on some interesting large scale geometric properties of these groups (see [Bur99], [BCS01], [BCST09]). However metric properties of Higman-Thompson's groups of type V are far less well understood, as it follows from the work of Birget [Bir04] that the number of carets is no longer quasi-isometric the the word-length in Thompson's group V .

Nevertheless, the following lemma gives a lower bound for the word metric in $V_{d,k}$ in terms of the number of carets. Note that the same result appears in [Bir04] for the case of Thompson's group V .

Proposition 3.6. *For any finite generating set Σ of $V_{d,k}$, there exists a constant $C_\Sigma > 0$ such that for any $v \in V_{d,k}$, we have $\kappa(v) \leq C_\Sigma |v|_\Sigma$.*

Proof. Define $C_\Sigma = \max_{\sigma \in \Sigma} \kappa(\sigma)$. Now remark that when multiplying, say on the right, an element $v \in V_{d,k}$ by an element $\sigma \in \Sigma$, we obtain an element $v\sigma$ having a canonical representative with trees having at most $\kappa(v) + C_\Sigma$ carets. This is because when expanding the domain tree of v to get a common expansion with the range tree of σ , we have to add at most C_Σ carets. So it follows from a straightforward induction that every element of length at most n

with respect to the word metric associated to Σ has a canonical representative with at most $C_\Sigma n$ carets, and the proof is complete. \square

4. THE ALMOST AUTOMORPHISM GROUPS $\text{AAut}_D(\mathcal{T}_{d,k})$

Almost automorphisms of $\mathcal{T}_{d,k}$ are homeomorphisms of the boundary $\partial_\infty \mathcal{T}_{d,k}$ which are piecewise tree automorphisms. In this section we introduce a family of subgroups of $\text{AAut}(\mathcal{T}_{d,k})$ consisting of almost automorphisms which are piecewise tree automorphisms of a given type.

4.1. Almost automorphisms of type $W(D)$. Let $D \leq \text{Sym}(d)$ be a subgroup of the symmetric group on d elements. Define recursively $D_1 = D$, seen as the subgroup of the automorphism group of the rooted d -regular tree \mathcal{T}_d acting on level one; and $D_{n+1} = D \wr D_n$ for every $n \geq 1$, where the permutational wreath product is associated with the natural action of D_n on the set of vertices of level n of \mathcal{T}_d . We now let $W(D)$ be the closed subgroup generated by the family (D_n) . For any $k \geq 1$ we let $W_k(D)$ be the subgroup of $\text{Aut}(\mathcal{T}_{d,k})$ fixing pointwise the first level and acting by an element of $W(D)$ in each subtree rooted at level one. The group $W_k(D)$ is naturally isomorphic to the product of k copies of the group $W(D)$.

Definition 4.1. An almost automorphism of $\mathcal{T}_{d,k}$ is said to be piecewise of type $W(D)$ if it can be represented by a triple (ψ, T, T') such that T, T' are finite rooted complete subtrees of $\mathcal{T}_{d,k}$ and $\psi : \mathcal{T}_{d,k} \setminus T \rightarrow \mathcal{T}_{d,k} \setminus T'$ belongs to $W(D)$ on each connected component, after the natural identification of each connected component of $\mathcal{T}_{d,k} \setminus T$ and $\mathcal{T}_{d,k} \setminus T'$ with \mathcal{T}_d .

We observe that by construction of $W(D)$, if a triple (ψ_1, T_1, T'_1) is such that $\psi_1 : \mathcal{T}_{d,k} \setminus T_1 \rightarrow \mathcal{T}_{d,k} \setminus T'_1$ belongs to $W(D)$ on each connected component, then for any equivalent triple (ψ_2, T_2, T'_2) such that T_2 (resp. T'_2) contains T_1 (resp. T'_1), then $\psi_2 : \mathcal{T}_{d,k} \setminus T_2 \rightarrow \mathcal{T}_{d,k} \setminus T'_2$ belongs to $W(D)$ on each connected component.

Proposition 4.2. *The set of almost automorphisms $\text{AAut}_D(\mathcal{T}_{d,k})$ which are piecewise of type $W(D)$ is a subgroup of $\text{AAut}(\mathcal{T}_{d,k})$.*

Proof. The only non-trivial fact that one needs to check is that $\text{AAut}_D(\mathcal{T}_{d,k})$ is closed under multiplication, but this follows from the previous observation and from the fact that $W(D)$ is a subgroup of $\text{Aut}(\mathcal{T}_d)$. \square

If D is the full permutation group $\text{Sym}(d)$ then $W(D) = \text{Aut}(\mathcal{T}_d)$ and $\text{AAut}_D(\mathcal{T}_{d,k}) = \text{AAut}(\mathcal{T}_{d,k})$. On the opposite, if D is the trivial group then being piecewise trivial means being locally order-preserving and $\text{AAut}_D(\mathcal{T}_{d,k}) = V_{d,k}$. It is straightforward from the definition that if D' contains D then $\text{AAut}_{D'}(\mathcal{T}_{d,k})$ contains $\text{AAut}_D(\mathcal{T}_{d,k})$. In particular we note that for every subgroup $D \leq \text{Sym}(d)$, the group $\text{AAut}_D(\mathcal{T}_{d,k})$ always contains $V_{d,k}$.

4.2. Topology on $\text{AAut}_D(\mathcal{T}_{d,k})$. By definition the group $\text{AAut}_D(\mathcal{T}_{d,k})$ also contains a copy of the tree automorphism group $W_k(D)$. The latter comes equipped with a natural group topology, which is totally disconnected and compact, defined by saying that the pointwise stabilizers of vertices of level n form a basis of neighbourhoods of the identity. We would like to extend

this topology to the group $\text{AAut}_D(\mathcal{T}_{d,k})$, i.e. define a group topology on $\text{AAut}_D(\mathcal{T}_{d,k})$ for which the subgroup $W_k(D)$ is an open subgroup. For, let us first recall the following well known lemma, a proof of which can be consulted in [Bou71, Chapter 3].

Lemma 4.3. *Let G be a group and let \mathcal{F} be a family of subgroups of G which is filtering, i.e. so that the intersection of any two elements of \mathcal{F} contains an element of \mathcal{F} . Assume moreover that for every $g \in G$ and every $U \in \mathcal{F}$, there exists $V \in \mathcal{F}$ so that $V \subset gUg^{-1}$. Then there exists a (unique) group topology on G for which \mathcal{F} is a base of neighbourhoods of the identity.*

Recall that a subgroup H of a group G is said to be commensurated by a subset K of G if for every $k \in K$, the subgroup $kHk^{-1} \cap H$ has finite index in both H and kHk^{-1} . The following easy lemma, whose proof is left to the reader, provides an easy way to check commensurability.

Lemma 4.4. *Let G be a group and S a generating set of G . Then a subgroup H of G is commensurated by G if and only if it is commensurated by S .*

Now let \mathcal{F} be the family of open subgroups of $W_k(D)$, which is a base of neighbourhoods of the identity in $W_k(D)$. It follows from Lemma 4.3 that if G is a group containing $W_k(D)$ as a subgroup, then there exists a group topology on G for which $W_k(D)$ is an open subgroup as soon as $W_k(D)$ is commensurated in G .

Now remark that Lemma 4.4 together with Proposition 4.8 (a corollary of which is that $\text{AAut}_D(\mathcal{T}_{d,k})$ is generated by $W_k(D)$ and $V_{d,k}$) imply that $W_k(D)$ is commensurated by $\text{AAut}_D(\mathcal{T}_{d,k})$, because it is trivially commensurated by itself and commensurated by $V_{d,k}$ by Lemma 4.7. We therefore obtain:

Proposition 4.5. *There exists a (unique) group topology on $\text{AAut}_D(\mathcal{T}_{d,k})$ turning $W_k(D)$ into a compact open subgroup. In particular $\text{AAut}_D(\mathcal{T}_{d,k})$ is a totally disconnected locally compact group (which is discrete if and only if $W_k(D)$ is trivial, if and only if D is trivial).*

Remark 4.6. 1) It is interesting to point out that whereas the topology on $\text{Aut}(\mathcal{T}_{d,k})$ coincides with the compact-open topology induced from $\text{Homeo}(\partial_\infty \mathcal{T}_{d,k})$, this is no longer true for the group $\text{AAut}(\mathcal{T}_{d,k})$. Indeed, the inclusion $\text{AAut}(\mathcal{T}_{d,k}) \hookrightarrow \text{Homeo}(\partial_\infty \mathcal{T}_{d,k})$ is continuous but has a non-closed image. In other words, the topology on $\text{AAut}(\mathcal{T}_{d,k})$ is strictly finer than the compact-open topology. Actually the image of $\text{AAut}(\mathcal{T}_{d,k}) \hookrightarrow \text{Homeo}(\partial_\infty \mathcal{T}_{d,k})$ is even dense, because one can check that the group $V_{d,k}$ is a dense subgroup of the homeomorphism group of $\partial_\infty \mathcal{T}_{d,k}$ with respect to the compact-open topology.

2) We also insist on the fact that for any permutation group D , the inclusion $\text{AAut}_D(\mathcal{T}_{d,k}) \hookrightarrow \text{AAut}(\mathcal{T}_{d,k})$ is always continuous, but its image is never closed unless D is the full permutation group $\text{Sym}(d)$. Indeed, $\text{AAut}_D(\mathcal{T}_{d,k})$ contains the subgroup $V_{d,k}$ which is dense in $\text{AAut}(\mathcal{T}_{d,k})$ by Remark 4.9, and therefore $\text{AAut}_D(\mathcal{T}_{d,k})$ is never closed inside $\text{AAut}(\mathcal{T}_{d,k})$ unless it is the whole group.

4.3. Preliminaries. In this section we establish preliminary results about the groups $\text{AAut}_D(\mathcal{T}_{d,k})$. Recall that $D \leq \text{Sym}(d)$ is a finite permutation group, and we define recursively a family of finite subgroups of $\text{Aut}(\mathcal{T}_d)$ by $D_1 = D$ and $D_{n+1} = D \wr D_n$ for every $n \geq 1$, where the permutational wreath product is associated with the natural action of D_n on the d^n vertices of level n of \mathcal{T}_d . We denote by D_∞ the subgroup generated by the family (D_n) (which also coincides with the increasing union of the family (D_n)) and by $W(D)$ the closure of D_∞ in $\text{Aut}(\mathcal{T}_d)$. For $n \in \{1, \dots, \infty\}$ we will also denote by D_n^k the subgroup of $W_k(D)$ fixing pointwise the first level of $\mathcal{T}_{d,k}$ and acting by an element of D_n on each subtree rooted at the first level. Note that these groups have a natural decomposition $D_n^k = D_n^{(1)} \times \dots \times D_n^{(k)}$, where $D_n^{(i)}$ is the subgroup of elements acting only on the i th subtree rooted at level one.

If $\sigma = (\psi, T, T') \in V_{d,k}$, we let $W_k(D)_\sigma$ be the subgroup of $W_k(D)$ consisting of automorphisms which are the identity on the subtree T . Note $W_k(D)_\sigma$ always contains some neighbourhood of the identity and is consequently an open subgroup of $W_k(D)$. The latter being compact, we obtain that $W_k(D)_\sigma$ is a finite index subgroup of $W_k(D)$.

Lemma 4.7. *For every $\sigma \in V_{d,k}$, we have the inclusion*

$$\sigma W_k(D)_\sigma \sigma^{-1} \subset W_k(D).$$

In particular $W_k(D)$ is commensurated by $V_{d,k}$.

Proof. If $\sigma = (\psi, T, T')$ and $u \in W_k(D)_\sigma$, the reader will easily check that the element $\sigma u \sigma^{-1} \in \text{AAut}_D(\mathcal{T}_{d,k})$ is represented by a triple (ψ', T', T') , where ψ' permutes trivially the connected components of $\mathcal{T}_{d,k} \setminus T'$. Now if we consider the tree automorphism $u' \in W_k(D)$ being the identity on T' and acting on $\mathcal{T}_{d,k} \setminus T'$ like ψ' , it is clear that u' is represented by the triple (ψ', T', T') , and therefore $\sigma u \sigma^{-1} = u' \in W_k(D)$. \square

The next result yields a decomposition of the group $\text{AAut}_D(\mathcal{T}_{d,k})$ in terms of the two subgroups $W_k(D)$ and $V_{d,k}$. It will be essential for proving Theorem 1.1 and Theorem 1.2.

Proposition 4.8. *For any $g \in \text{AAut}_D(\mathcal{T}_{d,k})$ there exists $(u, v) \in W_k(D) \times V_{d,k}$ such that $g = uv$.*

Proof. Let (ψ, T, T') be a triple representing $g \in \text{AAut}_D(\mathcal{T}_{d,k})$. Let us consider the element $v \in \text{AAut}_D(\mathcal{T}_{d,k})$ represented by the triple (ξ, T, T') where ξ is defined by declaring that each tree of the forest $\mathcal{T}_{d,k} \setminus T$ is globally sent on its image by ψ , but so that ξ is order-preserving on each connected component of $\mathcal{T}_{d,k} \setminus T$. Clearly we have $v \in V_{d,k}$. Now the default between g and v can be filled by performing the rooted tree automorphism induced by g on each subtree rooted at a leaf of T' . But all of these can be achieved at the same time by an element of $W_k(D)$, namely the automorphism being the identity on T' and acting as the desired rooted tree automorphism on each connected component of $\mathcal{T}_{d,k} \setminus T'$. \square

Remark 4.9. Actually in Proposition 4.8, $W_k(D)$ can be replaced by the pointwise stabilizer of the n th level of $\mathcal{T}_{d,k}$ in $W_k(D)$, for every $n \geq 1$, the

proof being the same. It yields in particular that $V_{d,k}$ is a dense subgroup of $\text{AAut}_D(\mathcal{T}_{d,k})$.

Now given $g \in \text{AAut}_D(\mathcal{T}_{d,k})$, there is not a unique $(u, v) \in W_k(D) \times V_{d,k}$ such that $g = uv$ because the two subgroups $W_k(D)$ and $V_{d,k}$ have a non-trivial intersection (as soon as D is non-trivial). The measure of how this decomposition fails to be unique naturally leads to the study of the intersection of these two subgroups.

Lemma 4.10. *The intersection between $V_{d,k}$ and $W_k(D)$ in $\text{AAut}_D(\mathcal{T}_{d,k})$ is D_∞^k .*

Proof. D_n^k lies inside $V_{d,k}$ and $W_k(D)$ for any $n \geq 1$, so the inclusion $D_\infty^k \subset V_{d,k} \cap W_k(D)$ is clear. To prove the converse inclusion, let g be an element of $V_{d,k} \cap W_k(D)$. Such an element g is an automorphism of $\mathcal{T}_{d,k}$ and therefore does act on the tree fixing setwise each level, so it is enough to prove that there exists an element of D_∞^k acting like g on $\mathcal{T}_{d,k}$. Since $g \in W_k(D)$, for every $n \geq 1$ there exists $g_n \in D_n^k$ acting like g on the first n levels of $\mathcal{T}_{d,k}$. But now since $g \in V_{d,k}$, it is eventually order-preserving and therefore $g = g_n$ for n large enough, which completes the proof. \square

The end of this section is devoted to establishing Lemma 4.11, which will be applied in the proof of Lemma 5.3 in Section 5. Roughly, the idea is to find a finite set of elements $\Delta \leq V_{d,k}$, so that given an element $u \in D_\infty^{(i)}$, we can find $\delta \in \Delta$ so that conjugating by δ increases by one the level of the action of u .

If $i = 1 \dots k$, recall that $\mathcal{T}_{d,k}^{a_i}$ denotes the full subtree of $\mathcal{T}_{d,k}$ rooted at a_i , and that (a_1, \dots, a_k) (resp. $(a_i b_1, \dots, a_i b_d)$) denotes the ordered vertices of level one of $\mathcal{T}_{d,k}$ (resp. $\mathcal{T}_{d,k}^{a_i}$). In what follows, by convention indexes will be taken modulo k (for example a_{k+1} will denote the vertex a_1).

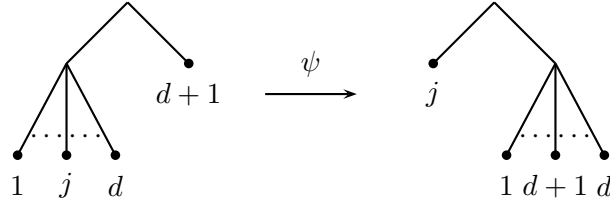
For every $i = 1 \dots k$ and $j = 1 \dots d$, we define an element $\delta_{i,j} = (\psi, T, T') \in V_{d,k}$ by the following manner:

- T is the smallest finite complete rooted subtree containing the d descendants of a_i ;
- T' is the smallest finite complete rooted subtree containing the d descendants of a_{i+1} ;
- ψ is defined by the formulas
 - $\psi(a_\ell) = a_\ell$ for every $\ell \notin \{i, i+1\}$;
 - $\psi(a_{i+1}) = a_{i+1} b_j$;
 - $\psi(a_i b_\ell) = a_{i+1} b_\ell$ for every $\ell \neq j$;
 - $\psi(a_i b_j) = a_i$.

For example the diagram of $\delta_{1,j}$ is represented in Figure 4.3 in the case $k = 2$. We denote by Δ the set of $\delta_{i,j}$, for $i = 1 \dots k$, $j = 1 \dots d$.

Lemma 4.11. *For any $i = 1 \dots k$, $j = 1 \dots d$, any $n \geq 1$ and any element $u \in D_\infty^{(i)}$ being an automorphism of $\mathcal{T}_{d,k}^{a_i b_j}$ with at most $n+1$ carets, the element $\delta_{i,j} u \delta_{i,j}^{-1}$ is an automorphism of $\mathcal{T}_{d,k}^{a_i}$ and has at most n carets.*

Proof. This is a direct consequence of the fact that $\delta_{i,j}$ maps the subtree $\mathcal{T}_{d,k}^{a_i b_j}$ to the the subtree $\mathcal{T}_{d,k}^{a_i}$. \square

FIGURE 2. The diagram of $\delta_{1,j}$ when $k = 2$.5. PRESENTATION OF $\text{AAut}_D(\mathcal{T}_{d,k})$

In this section we write down an explicit presentation of the group $\text{AAut}_D(\mathcal{T}_{d,k})$ for any $k \geq 1, d \geq 2$ and $D \leq \text{Sym}(d)$, and prove Theorem 1.1 and Theorem 1.2.

Let Σ denote a finite generating set of the group $V_{d,k}$, which is supposed to contain Δ . Enlarging Σ if necessary, we can also assume that Σ is saturated by Lemma 3.5. This implies the following:

Lemma 5.1. *We have the inclusion $\Sigma W_k(D) \subset W_k(D) \Sigma$.*

Proof. It follows from the proof of Proposition 4.8 that any $\sigma_1 u_1 \in \Sigma W_k(D)$ can be written $u_2 \sigma_2$ with $u_2 \in W_k(D)$ and $\sigma_2 \in V_{d,k}$ being of the form $(\psi, u^{-1}(T), T')$, where T is the domain tree of σ . Since Σ is saturated, σ_2 belongs to Σ and therefore $\sigma_1 u_1 \in W_k(D) \Sigma$. \square

According to Proposition 4.8, the set $S = \Sigma \cup W_k(D)$ is a generating set of $\text{AAut}_D(\mathcal{T}_{d,k})$. The strategy to prove Theorem 1.1 will be to list some particular relations between the elements of S satisfied in the group $\text{AAut}_D(\mathcal{T}_{d,k})$, and then to prove that they generate all the relations in $\text{AAut}_D(\mathcal{T}_{d,k})$.

- (R_Σ) According to Theorem 3.3 there exists a finite set of words $R_\Sigma \leq \Sigma^*$ so that $\langle \Sigma \mid R_\Sigma \rangle$ is a presentation of $V_{d,k}$.
- (R_D) We let R_D be the set of words of the form $u_1 u_2 u_3^{-1}$, $u_i \in W_k(D)$, whenever the relation $u_1 u_2 = u_3$ is satisfied in the group $W_k(D)$.
- (R_1) The set of relations R_1 will correspond to commensurating relations in $\text{AAut}_D(\mathcal{T}_{d,k})$. Recall that if $\sigma \in V_{d,k}$ and $u \in W_k(D)_\sigma$ then $\sigma u \sigma^{-1} \in W_k(D)$ by Lemma 4.7. We let R_1 be the set of words of the form $\sigma u_1 \sigma^{-1} u_2^{-1}$, where $\sigma \in \Sigma, u_1 \in W_k(D)_\sigma, u_2 \in W_k(D)$, whenever the relation $\sigma u_1 \sigma^{-1} = u_2$ holds in $\text{AAut}_D(\mathcal{T}_{d,k})$.
- (R_2) We add relations corresponding to the fact that the subgroup D_1^k of $\text{AAut}_D(\mathcal{T}_{d,k})$ lies in the intersection of $V_{d,k}$ and $W_k(D)$. More precisely, for every $i \in \{1, \dots, k\}$ and every $u \in D_1^{(i)}$, we choose a word $w_u \in \Sigma^*$ so that $u = w_u$ in $\text{AAut}_D(\mathcal{T}_{d,k})$. We denote by R_2 the set of words $u w_u^{-1}$, and by r_i the maximum word length of the words w_u when u ranges over $D_1^{(i)}$.
- (R_3) By Lemma 5.1, for every $\sigma_1 \in \Sigma$ and $u_1 \in W_k(D)$ we can pick some $u_2 \in W_k(D)$ and $\sigma_2 \in \Sigma$ so that $\sigma_1 u_1 = u_2 \sigma_2$ in $\text{AAut}_D(\mathcal{T}_{d,k})$. We denote by R_3 the set of words $\sigma_1 u_1 \sigma_2^{-1} u_2^{-1}$.

Denote by $R = R_\Sigma \cup R_D \cup_i R_i$ the union of all these relations. Note that elements of R have bounded length with respect to the compact generating set $S = \Sigma \cup W_k(D)$ of $\text{AAut}_D(\mathcal{T}_{d,k})$.

We let G be the group defined by the presentation $\langle S \mid R \rangle$, that is we have a short exact sequence

$$1 \rightarrow \mathcal{R} = \langle\langle R \rangle\rangle \rightarrow F_S \rightarrow G \rightarrow 1,$$

where F_S is the free group over the set S and $\langle\langle R \rangle\rangle$ is the normal subgroup generated by R . Denote by $a : F_S \rightarrow [0, +\infty]$ the corresponding area function, which by definition associates to $w \in \mathcal{R}$ the least integer n so that w is a product of at most n conjugates of elements of R , and $a(w) = +\infty$ if $w \notin \mathcal{R}$. We also define the associated cost function $c : F_S \times F_S \rightarrow [0, +\infty]$ by $c(w_1, w_2) = a(w_1^{-1}w_2)$. This function estimates the cost of converting w_1 to w_2 , or the cost of going from w_1 to w_2 , in the sense that $c(w_1, w_2)$ is the distance in F_S between w_1 and w_2 with respect to the word metric associated to the union of conjugates of R . In particular the cost function is symmetric and satisfies the triangular inequality $c(w_1, w_3) \leq c(w_1, w_2) + c(w_2, w_3)$ for every $w_1, w_2, w_3 \in F_S$. This, combined with the bi-invariance of the cost function, yields the following inequality, which will be used repeatedly: for every $\ell \geq 1$ and every $w_1, \dots, w_\ell, w'_1, \dots, w'_\ell \in F_S$, we have:

$$c(w_1 \dots w_\ell, w'_1 \dots w'_\ell) \leq \sum_{i=1}^{\ell} c(w_i, w'_i).$$

Two words $w_1, w_2 \in F_S$ are said to be homotopic if they represent the same element of G , i.e. if $c(w_1, w_2) < +\infty$. A word w is said to be null-homotopic if it represents the identity, i.e. if $w \in \mathcal{R}$.

We are now able to state the main theorem of this section, which implies both Theorem 1.1 and Theorem 1.2.

Theorem 5.2. *The natural map $G \rightarrow \text{AAut}_D(\mathcal{T}_{d,k})$ is an isomorphism. Furthermore, the Dehn function of the presentation $\langle S \mid R \rangle$ is asymptotically bounded by that of $V_{d,k}$.*

It is clear that the map from G to $\text{AAut}_D(\mathcal{T}_{d,k})$ is a well defined morphism because relations $R_\Sigma, R_D, (R_i)$ are satisfied in $\text{AAut}_D(\mathcal{T}_{d,k})$, and it is onto because S generates the group $\text{AAut}_D(\mathcal{T}_{d,k})$. So proving the first claim comes down to proving that this morphism is injective, i.e. any word in F_S representing the identity in the group $\text{AAut}_D(\mathcal{T}_{d,k})$ already represents the trivial element in the group G . This will be achieved, as well as the proof of the upper bound on the Dehn function, in Proposition 5.6, using both geometric and combinatorial arguments.

The goal of Lemma 5.3 and Corollary 5.4 is to prove that relations in the group $\text{AAut}_D(\mathcal{T}_{d,k})$ coming from the fact that the subgroups $W_k(D)$ and $V_{d,k}$ intersect non-trivially, are already satisfied in the group G , and to obtain a precise estimate of their cost.

Lemma 5.3. *Fix $i \in \{1, \dots, k\}$ and let $C_i = 2d + \max(2, r_i)$ (recall that r_i has been defined with the set of relators R_2). Then for every $n \geq 0$ and every*

$u \in D_\infty^{(i)}$ having at most n carets, there exists a word $w \in \Sigma^*$ of length at most $C_i n$ so that the relation $u = w$ holds in G and has cost at most $C_i n$.

Proof. We use induction on n . The result is trivially true for $n = 0$ because the only element of $D_\infty^{(i)}$ with zero caret is the identity, and is true for $n = 1$ thanks to the set of relators R_2 .

The idea of the proof of the induction step is the following. Given $u \in D_\infty^{(i)}$ with at most $n + 1$ carets, we begin by multiplying it by an element of $D_1^{(i)}$ in order to ensure that it acts trivially on the first level of $\mathcal{T}_{d,k}^{a_i}$. The resulting automorphism has a natural decomposition into a product of d elements of $D_\infty^{(i)}$, coming from its action on the subtrees $\mathcal{T}_{d,k}^{a_i b_1}, \dots, \mathcal{T}_{d,k}^{a_i b_d}$, with a nice control on the number of carets of each element of this product. We then apply the induction hypothesis to each of these elements, after having reduced their number of carets by conjugating by an element of Δ , which has the effect of increasing by 1 the level of the subtree on which they act.

Henceforth we assume that $u \in D_\infty^{(i)}$ is an element having at most $n + 1$ carets, with $n \geq 1$. If we let \bar{u} denote the element of $D_1^{(i)}$ acting like u on the first level of $\mathcal{T}_{d,k}^{a_i}$, it is clear that $u' = u\bar{u}^{-1}$ stabilizes pointwise the first level of $\mathcal{T}_{d,k}^{a_i}$. Using relators from R_2 , we pick a word $w_{\bar{u}} \in \Sigma^*$ so that the relation $\bar{u} = w_{\bar{u}}$ holds in G and has cost at most one.

Now in the group $\text{AAut}_D(\mathcal{T}_{d,k})$, since u' acts trivially on the first level of $\mathcal{T}_{d,k}^{a_i}$, it has a natural decomposition $u' = u_1 \dots u_d$, where each $u_\ell \in D_\infty^{(i)}$ acts on the subtree of $\mathcal{T}_{d,k}^{a_i b_\ell}$. Note that each u_ℓ has at most $n + 1$ carets and that $\sum_\ell \kappa(u_\ell) \leq \kappa(u) + d - 1 \leq n + d$, because the caret corresponding to the root of $\mathcal{T}_{d,k}^{a_i}$ can appear d times in this sum, whereas it is counted only once in $\kappa(u)$. Note also that thanks to the set of relators R_D , the relation $u' = u_1 \dots u_d$ also holds in the group G and has cost at most d .

Remark that by construction of the set Δ , every element of $W_k(D)$ acting trivially on the second level of $\mathcal{T}_{d,k}$ lies inside $W_k(D)_\delta$ for every $\delta \in \Delta$. In particular if $\ell \in \{1, \dots, d\}$ and if $\delta_\ell = \delta_{i,\ell}$, we have $u_\ell \in W_k(D)_{\delta_\ell}$ and thanks to R_1 , the word $\delta_\ell u_\ell \delta_\ell^{-1}$ represents in the group G an element $\tilde{u}_\ell \in D_\infty^{(i)}$ with at most $\kappa(u_\ell) - 1 \leq n$ carets according to Lemma 4.11. Note in particular that

$$(1) \quad \sum_\ell \kappa(\tilde{u}_\ell) \leq \sum_\ell (\kappa(u_\ell) - 1) \leq \sum_\ell \kappa(u_\ell) - d \leq n.$$

For every $\ell \in \{1, \dots, d\}$, we now apply the induction hypotheses to \tilde{u}_ℓ and obtain a word \tilde{w}_ℓ of length at most $C_i \kappa(\tilde{u}_\ell)$ so that $\tilde{u}_\ell = \tilde{w}_\ell$ in G and $c(\tilde{u}_\ell, \tilde{w}_\ell) \leq C_i \kappa(\tilde{u}_\ell)$. If we denote by $w_\ell = \delta_\ell^{-1} \tilde{w}_\ell \delta_\ell$, then the relation $u_\ell = w_\ell$ holds in the group G and has cost

$$c(u_\ell, w_\ell) \leq c(u_\ell, \delta_\ell^{-1} \tilde{u}_\ell \delta_\ell) + c(\delta_\ell^{-1} \tilde{u}_\ell \delta_\ell, w_\ell) \leq 1 + C_i \kappa(\tilde{u}_\ell).$$

We now want to put all these pieces together and conclude the proof of the induction step. For, let $w = w_1 \dots w_d w_{\bar{u}} \in \Sigma^*$. Its length easily satisfies

$$|w|_\Sigma \leq \sum_{\ell=1}^d |w_\ell|_\Sigma + |w_{\bar{u}}|_\Sigma \leq \sum_{\ell=1}^d (2 + |\tilde{w}_\ell|_\Sigma) + r_i \leq C_i \sum_{\ell=1}^d \kappa(\tilde{u}_\ell) + 2d + r_i \leq C_i(n+1),$$

because $\sum_{\ell} \kappa(\tilde{u}_{\ell}) \leq n$ according to (1), and $C_i \geq 2d + r_i$. Furthermore, we claim that the relation $u = w$ is satisfied in G and has cost at most $C_i(n+1)$, which follows from the following summation of cost estimates:

$$\begin{aligned}
c(u, w) &\leq c(u, u'\bar{u}) + c(u'\bar{u}, w) \\
&\leq 1 + c(u', w_1 \dots w_d) + c(\bar{u}, w_{\bar{u}}) \\
&\leq 1 + c(u', u_1 \dots u_d) + c(u_1 \dots u_d, w_1 \dots w_d) + 1 \\
&\leq 2 + d + \sum c(u_{\ell}, w_{\ell}) \\
&\leq 2 + d + \sum (1 + C_i \kappa(\tilde{u}_{\ell})) \\
&\leq 2 + 2d + C_i \sum \kappa(\tilde{u}_{\ell}) \\
&\leq 2 + 2d + C_i n \\
&\leq C_i(n+1),
\end{aligned}$$

so the proof of the induction step is complete. \square

Corollary 5.4. *There exists a constant $C > 0$ such that for every $u \in D_{\infty}^k$, there exists a word $w \in \Sigma^*$ of length at most $C\kappa(u)$ so that the relation $u = w$ holds in G and has cost at most $k + \kappa(u)$.*

Proof. Let $C = \max_i C_i$, where the constant C_i is defined in Lemma 5.3. Any $u \in D_{\infty}^k$ can be written $u = u_1 \dots u_k$ in $\text{AAut}_D(\mathcal{T}_{d,k})$, with $u_i \in D_{\infty}^{(i)}$ and $\kappa(u) = \kappa(u_1) + \dots + \kappa(u_k)$. Applying Lemma 5.3 to u_i , we get a word w_i of length at most $C_i \kappa(u_i)$ so that the relation $u_i = w_i$ holds in G and has cost at most $\kappa(u_i)$. Let $w = w_1 \dots w_k$. Then

$$|w|_{\Sigma} \leq \sum_{i=1}^k |w_i|_{\Sigma} \leq \sum_{i=1}^k C_i \kappa(u_i) \leq C \sum_{i=1}^k \kappa(u_i) = C\kappa(u).$$

Moreover the relation $u = u_1 \dots u_k$ holds in G thanks to the set of relators R_D . Consequently in G we have $u = w$ at a total cost of at most

$$\begin{aligned}
c(u, u_1 \dots u_k) + c(u_1 \dots u_k, w) &\leq k + \sum_{i=1}^k c(u_i, w_i) \\
&\leq k + \sum_{i=1}^k \kappa(u_i) = k + \kappa(u).
\end{aligned}$$

\square

The next lemma will reduce the estimate of the area function to its estimate for words of the special form $W_k(D)\Sigma^*$.

Lemma 5.5. *There exists a constant $c_1 > 0$ such that for any n and any word $w \in S^*$ of length at most n , there exists a word $w' = u\sigma_1 \dots \sigma_j$ of length at most n , where $u \in W_k(D)$, $\sigma_1, \dots, \sigma_j \in \Sigma$, so that w' is homotopic to w and $c(w, w') \leq c_1 n \log(n)$.*

Proof. For any word $w \in S^*$, define

$$\tau(w) = \inf \{c(w, w') : w' \in W_k(D)\Sigma^* \text{ and } w' \text{ is homotopic to } w\},$$

and

$$f(n) = \sup \{ \tau(w) : w \in S^* \text{ has length at most } n \}.$$

Note that both τ and f take finite values thanks to relators from R_3 and R_D . We want to prove that $f(n) \leq c_1 n \log(n)$ for some constant c_1 .

We use an algorithmic strategy. Given a word w , we first divide it into two subwords, then apply the algorithm to each of them and finally merge the results. More precisely, let us consider a word w of length 2^{n+1} , and divide it into two subwords w_1, w_2 of length 2^n . By definition of the function f , there exists words $w'_1, w'_2 \in W_k(D)\Sigma^*$ such that $c(w_1, w'_1), c(w_2, w'_2) \leq f(2^n)$. Now in the word $\bar{w} = w'_1 w'_2 \in W_k(D)\Sigma^* W_k(D)\Sigma^*$ we can move the $W_k(D)$ part of w'_2 to the left by applying at most $2^n - 1$ relators of R_3 , and merge it with the $W_k(D)$ part of w'_1 with cost 1 thanks to the set of relators R_D . We therefore get a word $w' \in W_k(D)\Sigma^*$ homotopic to w and so that $c(w, w') \leq 2f(2^n) + (2^n - 1) + 1$, which implies that $\tau(w) \leq 2f(2^n) + 2^n$. By definition of f , we obtain $f(2^{n+1}) \leq 2f(2^n) + 2^n$, from which we easily get the inequality $f(2^n) \leq n2^{n-1}$. The result then follows from this inequality together with the fact that f is non-decreasing. \square

Proposition 5.6. *There exists a constant $c > 0$ such that if $w \in F_S$ is a null-homotopic word of length at most n in $\text{AAut}_D(\mathcal{T}_{d,k})$, then w already represents the identity in G and has area*

$$a(w) \leq cn \log(n) + \delta(cn),$$

where δ is the Dehn function of the presentation $\langle \Sigma, R_\Sigma \rangle$ of $V_{d,k}$.

Proof. We first apply Lemma 5.5 to w and get a word $w' = u\sigma_1 \dots \sigma_j$ so that $c(w, w') \leq c_1 n \log(n)$. Since w is null-homotopic, so is w' and therefore the element u^{-1} belongs to $W_k(D) \cap V_{d,k} = D_\infty^k$, and has length at most n in the group $V_{d,k}$ because w' has length at most n . According to Proposition 3.6, we have $\kappa(u^{-1}) \leq C_\Sigma n$. Applying Corollary 5.4 to u^{-1} yields a word $w'' \in \Sigma^*$ of length at most $C\kappa(u^{-1}) \leq CC_\Sigma n$ so that the relation $u^{-1} = w''$ holds in G and has cost at most $k + C_\Sigma n$. Therefore we obtain that

$$\begin{aligned} a(w) &\leq c(w, w') + c(u^{-1}, w'') + c(w'', \sigma_1 \dots \sigma_j) \\ &\leq c_1 n \log(n) + (k + C_\Sigma n) + \delta(CC_\Sigma n + n), \end{aligned}$$

because w'' and $\sigma_1 \dots \sigma_j$ represents the same element in $V_{d,k}$ and $w''(\sigma_1 \dots \sigma_j)^{-1}$ has length at most $CC_\Sigma n + n$, so $c(w'', \sigma_1 \dots \sigma_j)$ is at most $\delta(CC_\Sigma n + n)$ by definition of the Dehn function. Therefore $a(w) \leq cn \log(n) + \delta(cn)$ for some constant c depending only on Σ , and the proof is complete. \square

In particular we deduce from Proposition 5.6 that the Dehn function of $\text{AAut}_D(\mathcal{T}_{d,k})$ is $\preceq n \log n + \delta_{V_{d,k}}$. But now the group $V_{d,k}$ is not Gromov-hyperbolic since it has a \mathbb{Z}^2 subgroup, so its Dehn function is not linear and consequently at least quadratic [Bow95]. Therefore $n \log n \preceq \delta_{V_{d,k}}$, and the Dehn function of $\text{AAut}_D(\mathcal{T}_{d,k})$ is thus asymptotically bounded by $\delta_{V_{d,k}}$.

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